Qualifying Exam (Spring 2021): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10,11,12,13,14

All problems are weighted equally. On this cover page write which seven problems you want graded.

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

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1). Answer the following three parts:

(a) Give the dual of the problem

$$\begin{cases} (LP) & \min & \mathbf{c}^T\mathbf{x} \\ & \text{st} & A\mathbf{x} = \mathbf{b} \\ & & 0 \le \mathbf{x} \le \mathbf{u} \end{cases}$$

(b) Show that the dual of LP is always feasible.

(c) Suppose you found a feasible solution for LP. What can you conclude?

2). Consider the following (knapsack) ILP:

$$\begin{array}{rcccc} \max\ z & = & 150x_1 + 100x_2 + 99x_3 & & \\ & & 51x_1 + 50x_2 + 50x_3 & \le & 100 \\ & & x_1, x_2, x_3 & \ge & 0 \\ & & x_1, x_2, x_3 & & \text{integer} \end{array}$$

Solve this problem by first finding the LP optimum and then finding the integer optimum by the cutting plane method.

3). Consider the cost matrix in the table for a transportation problem in which the objective is to minimize cost (Rows: sources; Columns: destinations).

	Destination			
Source	1	2	3	Supply
1	8	5	4	50
2	6	8	9	20
Demand	10	20	40	

(a) Write the Linear Programming formulation for this problem.

(b) Set up the transportation tableau and use the Northwest Corner Rule to find an initial BFS.

(c) Begining with the initial solution found in part (b), solve the problem using the transportation simplex method. Give an optimal primal solution and an optimal dual solution.

(d) Write the dual of the LP formulated in part (a). Verify that the dual solution found in part (c) is feasible to the dual problem.

4). Let $\{N(t), t \ge 0\}$ be a Poisson process with arrival rate λ. A Bernoulli splitting procedure is used to split the process into r processes, i.e., $N(t) = N_1(t) + N_2(t) + \ldots + N_r(t)$ for all $t \ge 0$. Assume that a random arrival in $\{N(t)\}$ is recorded by $\{N_i(t)\}$ with a constant probability $p_i \in (0,1)$, $i = 1, \ldots, r$, where p_i's satisfy $\sum_{i=1}^r p_i = 1$. Prove that (a) $\{N_i(t)\}$ is a Poisson process with arrival rate λp_i; (b) for a given time t, $N_1(t), \ldots, N_r(t)$ are independent random variables.

5). Let $\{B(t), t \ge 0\}$ be the excess life process associated with a renewal process $\{N(t), t \ge 0\}$, i.e., $B(t) = S_{N(t)+1} - t$ for all t, where S_n is the arrival time of the nth event in $\{N(t)\}$. Denote by $G(\cdot)$ and μ the respective c.d.f. and mean of the sequence of interarrival times. For a given $x > 0$, compute the long run probability $\lim_{t\to\infty} P(B(t) > x)$.

6). Consider an $M/M/1/K$ queueing system with Poisson arrivals $PP(\lambda)$ and i.i.d exponential service times with service rate μ. Suppose at time $t = 0$ there is a single customer in the system. Let S be the arrival time of the first customer who sees the system empty. Compute $E[S]$.

7). Give an algorithm for generating random variates from the following cumulative distribution function:

$$F(x) = \begin{cases} \frac{1-e^{-2x}+2x}{3}, & \text{if } 0 < x \leq 1 \\ \frac{3-e^{-2x}}{3}, & \text{if } 1 < x < \infty. \end{cases}$$

8). Let X and Y be two independent random variables with respective cumulative distribution functions $F(x)$ and $G(y)$. Suppose we have generated n independent random variates $X_1, \ldots, X_n$ from $F(x)$ and n independent random variates $Y_1, \ldots, Y_n$ from $G(y)$. Give an unbiased estimator (based on $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$) for estimating the probability $P(X < Y)$.

9). Consider a Markov decision process with a finite state space X, a finite action set A, sets of feasible actions $A(x) \subset A$ at states x, one-step rewards $r(x, a)$, and one-step transition probabilities $p(y|x, a)$ from states x to states y, where $x, y \in X$ and $a \in A(x)$. The goal is to find a policy π maximizing average expected costs per unit time

$$w^\pi(x) = \liminf_{N \to \infty} \frac{1}{N} E_x^\pi \sum_{t=0}^{N-1} r(x_t, a_t), \qquad x \in X.$$

Prove that there exists a nonrandomized stationary optimal policy. We recall that a nonrandomized stationary policy (sometimes called deterministic or stationary) is defined by a function $\phi : X \mapsto A$ such that $\phi(x) \in A(x)$ for all $x \in X$.

10). Consider a Markov decision process with a countable state space X, a finite action set A, sets of feasible actions $A(x) \subset A$ at states x, nonnegative one-step rewards $r(x, a)$, and one-step transition probabilities $p(y|x, a)$ from states x to states y, where $x, y \in X$ and $a \in A(x)$. The goal is to find a policy π maximizing expected total costs

$$w^\pi(x) = E_x^\pi \sum_{t=0}^{\infty} r(x_t, a_t), \qquad x \in X.$$

Provide an example when an optimal policy does not exist even if $w^\pi(x) < \infty$ for every policy π and for every initial state $x \in X$.

11). Let $(X_j)_{j \geq 1}$ be i.i.d. with X_j in L^1. Let $Y_j = e^{X_j}$. Show that

$$\left(\prod_{j=1}^{n} Y_j \right)^{\frac{1}{n}}$$

converges to a constant α a.s. and find α.

12). For a probability space $(\Omega, \mathcal{A}, P)$, for $X, Y \in L^2(\Omega, \mathcal{A}, P)$, and for a σ-algebra $\mathcal{G} \subset \mathcal{A}$, prove the Cauchy-Schwartz inequality

$$(E\{XY|\mathcal{G}\})^2 \leq E\{X^2|\mathcal{G}\}E\{Y^2|\mathcal{G}\}.$$

13). Let $S = \{t_1, \ldots, t_n\}$ be a set of n triangles in the plane, in general position (no 3 triangle vertices are collinear), with no horizontal triangle edges and no vertical triangle edges. Recall that we consider a triangle to be a *closed* region, including its boundary and its interior.

(a). How efficiently (in big-Oh) can one determine if the n triangles S are disjoint. (Not only are the boundaries disjoint, but there is no triangle contained within another triangle's interior.) Justify.

(b). We say that S is in *convex position* if every triangle t_i has at least one of its vertices as an extreme point of the convex hull of S. How efficiently (in big-Oh) can one determine if S is in convex position? Give the best bound you can and explain briefly.

(c). How efficiently (in big-Oh) can we determine if the convex hull of S is a hexagon? Explain briefly.

(d). How efficiently (in big-Oh) can we determine if the intersection of all triangles, $t_1 \cap t_2 \cap \cdots \cap t_n$, is nonempty? (i.e., if there exists a point that lies inside of *all* n triangles) Justify briefly (without any algorithmic details).

(e). Assume now that the n triangles in S are pairwise disjoint. Suppose we want to preprocess S for the following type of query very efficiently: Given a query point q, does q see the origin (point (0,0)), when the triangles t_i are considered to be obstacles? (i.e., does the line segment from the origin to point q intersect any of the triangles S?) What preprocessing/space/query time can you achieve for this? Explain briefly (without algorithmic details).

14). Let P be a simple n-gon in the plane.

(a). Suppose G is a *minimal vertex guard set* within P: G is a set of vertices of P so that every point of P is seen by at least one point (vertex) of G (i.e., G is a valid vertex guard cover of P), *and* the set G is *minimal*, meaning that deletion of any one point from G will cause G to stop being a valid guard cover of P). Give an example showing that G can have at least 5 times as many points as has a *minimum* vertex guard cover G^* (a set of $g_V(P)$ vertices that is a valid guard cover of P and has the fewest points of any guard cover of P using vertices of P).

(b). How efficiently can one compute a set G of at most $n/2$ vertices of P so that G is a valid guard cover of P?

(c). The following algorithm has been proposed to compute a set G of at most $n/2$ vertices of P so that G is a valid guard cover of P: Consider the ordered (ccw) list of vertices, $(v_1, v_2, \ldots, v_n)$, of P, and place guards at the odd-index vertices, $v_1, v_3, v_5, \ldots$. (The "first" vertex, v_1, is specified and given to us; it can be any vertex of P.) Does the algorithm work (to give a valid guard set of at most $n/2$ vertex guards)? If yes, explain briefly why; if no, give a counterexample.

(d). Suppose now that our goal is to find a set of diagonals of P that decompose P into a small number of convex polygons (e.g., so that we can place one guard within each convex polygon, thereby getting a valid guard cover). Let OPT denote the minimum possible number of such convex pieces in a decomposition of P by diagonals. What is an efficient way to obtain an approximation algorithm for computing OPT? How good is this approximation and what is its running time? Explain very briefly.